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Differential equations for multiple orthogonal polynomials with respect to classical weights: raising and lowering operators

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Abstract

We obtain a lowering operator for multiple orthogonal polynomials having orthogonality conditions with respect to $r \in \mathbb{N}$ classical weights. These multiple orthogonal polynomials are generalizations of the classical orthogonal polynomials. Combining the lowering operator with the raising operators, which have been obtained earlier in the literature, we then also obtain a linear differential equation of order $r + 1$.

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1. Introduction

In this paper, we study some examples of multiple orthogonal polynomials [2, 12] of type II. These polynomials are a generalization of orthogonal polynomials [10]. They arise naturally in the theory of simultaneous rational approximation, in particular as common denominators in Hermite–Padé approximation of type II to $r \in N$ (Markov) functions, which has its roots in the 19th century. Some of their applications are situated in diophantine number theory, rational approximation, spectral and scattering problems for higher order difference equations and some associated dynamical systems, see, e.g., [2, 3, 9, 11, 13, 14]. Recently they have also appeared in random matrix theory for matrix ensembles with external source [6–8] and Wishard ensembles [5].

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\vec{n} = (n_1, n_2, \ldots, n_r) \in \mathbb{N}_0^r$ be a vector of r non-negative integers, which is called a *multi-index* with length $|\vec{n}| := n_1 + n_2 + \cdots + n_r$. A multiple orthogonal polynomial $P_{\vec{n}}$ of type II with respect to the multi-index $\vec{n}$ and the system of r weights w_j on contours Γ_j, $j = 1, \ldots, r$, is then a (nontrivial) polynomial of degree $\leqslant |\vec{n}|$ which satisfies the orthogonality conditions

$$\int_{\Gamma_j} P_{\vec{n}}(z) z^m w_j(z)\, \mathrm{d}z = 0, \qquad 0 \leqslant m \leqslant n_j - 1, \quad j = 1, \ldots, r. \tag{1.1}$$

Table 1. Different singularities at zero.

Case	$\{w_j(z) = w^{\alpha_j,\beta}(z) = z^{\alpha_j}\omega^{\beta}(z)\}_{j=1}^{r}$	Γ	μ	ν	$A(\alpha_j, \beta)$
Jacobi–Piñeiro	$z^{\alpha_j}(z-1)^{\beta}, \alpha_j, \beta > -1$	$(0, 1)$	1	1	$\alpha_j + \beta + 2$
Laguerre I	$z^{\alpha_j}\,\mathrm{e}^{\beta z}, \alpha_j > -1, \beta < 0$	$(0, +\infty)$	1	0	β

Table 2. Different exponential rates at infinity.

Case	$\{w_j(z) = w^{\alpha,\beta_j}(z) = \upsilon^{\alpha}(z)\,\mathrm{e}^{\beta_j z}\}_{j=1}^{r}$	Γ	μ	ν	$A(\alpha, \beta_j)$
Laguerre II	$z^{\alpha}\,\mathrm{e}^{\beta_j z}, \alpha > -1, \beta_j < 0$	$(0, +\infty)$	1	0	β_j
Hermite	$\mathrm{e}^{\frac{\alpha}{2}z^2+\beta_j z}, \alpha < 0$	$(-\infty, +\infty)$	0	0	α

A multi-index $\vec{n}$ is called *normal* if all the solutions of (1.1) have exactly degree $|\vec{n}|$, implying uniqueness up to a multiplicative normalization constant. We will deal with *perfect* systems of weights, meaning that all the multi-indices are normal. Some famous classes of perfect systems are the Angelesco systems, some Nikishin systems and the AT systems, see, e.g., [12, 14].

For the examples in tables 1 and 2 the existence of a linear differential equation of order $r + 1$ was proven in [4]. The authors also showed that the polynomial coefficients can be computed recursively. Our aim is to present an easier way to derive the linear differential equation of order $r + 1$ based on raising and lowering operators. In [15] this idea was already applied to some discrete examples of multiple orthogonal polynomials (multiple Charlier, multiple Meixner I and II) in the case $r = 2$. Moreover, in [11] this was done for multiple Hermite polynomials.

The examples in tables 1 and 2 are AT systems for which $\Gamma_1 = \cdots = \Gamma_r = \Gamma$. Moreover, all the weights w_j belong to the same *classical* two-parameter family $\{w^{\alpha,\beta}\}$, changing only one of the parameters. In particular, they satisfy

(P1) Pearson's equation

$$\frac{\mathrm{d}}{\mathrm{d}z}(\phi(z)w^{\alpha,\beta}(z)) = \psi(z;\alpha,\beta)w^{\alpha,\beta}(z) \tag{1.2}$$

with ϕ, $\psi(\cdot;\alpha,\beta)$ polynomials for which $\deg(\phi) \leqslant 2$ and $\deg(\psi(\cdot;\alpha,\beta)) = 1$.

Furthermore, we observe that they have the following more technical properties which we will use throughout this article:

(P2) $\phi(z) = z^{\mu}(z-1)^{\nu}$, $\mu, \nu \in \mathbb{N}_0$, $\deg(\phi) = \mu + \nu \leqslant 2$.
(P3) $A(\alpha,\beta) + n\delta_{\deg(\phi),2} \neq 0$, $n \in \mathbb{N}_0$, where $\psi(z;\alpha,\beta) = A(\alpha,\beta)z + B(\alpha,\beta)$.
(P4) $\phi(z)w^{\alpha,\beta}(z) = w^{\alpha+\mu,\beta+\nu}(z)$.
(P5) $\Delta_{\Gamma}\left(w^{\alpha+\mu,\beta+\nu}(z)z^m\right) = 0$, $m \in \mathbb{N}$, where $\Delta_{\Gamma} f(z)$ indicates the difference of f at the endpoints of Γ.

These properties will provide the raising and lowering operators for the corresponding multiple orthogonal polynomials of type II. The raising operators were mentioned in the literature before, see, e.g., [14]. A combination of them then gives their differential equation of order $r + 1$.

1.1. Table 1

In the examples of table 1 the r classical weights have the form

$$w^{\alpha_j,\beta}(z) = z^{\alpha_j}\omega^{\beta}(z), \qquad j = 1, \ldots, r \tag{1.3}$$

with $\alpha_j - \alpha_k \notin \mathbb{Z}$, $j \neq k$. So they are of the same type, but with different singularities at zero. As mentioned in, e.g., [14] the corresponding multiple orthogonal polynomials $P_{\vec{n}}^{\vec{\alpha};\beta}$ have the following *raising operators*:

$$\frac{1}{w^{\alpha_j,\beta}(z)}\frac{\mathrm{d}}{\mathrm{d}z}\left(w^{\alpha_j+1,\beta+\nu}(z)P_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j;\beta+\nu}(z)\right) = P_{\vec{n}}^{\vec{\alpha};\beta}(z), \qquad n_j > 0, \tag{1.4}$$

$j = 1, \ldots, r$, with $\vec{e}_j$ the jth standard unit vector and $\vec{e} = (1, \ldots, 1) \in \mathbb{R}^r$. Here we mention that $\mu = 1$ for the examples in table 1. The proof of (1.4) is then based on integration by parts together with the properties (P1)–(P5), where the correct degree is provided by (P3). Note that relations (1.4) fix the normalization of the polynomials $P_{\vec{n}}^{\vec{\alpha};\beta}$. Repeatedly applying these raising operators gives the *Rodrigues formula*

$$P_{\vec{n}}^{\vec{\alpha};\beta}(z) = \frac{1}{\omega^{\beta}(z)}\left[\prod_{j=1}^{r}\left(z^{-\alpha_j}\frac{\mathrm{d}^{n_j}}{\mathrm{d}z^{n_j}}z^{\alpha_j+n_j}\right)\right](\omega^{\beta+\nu|\vec{n}|}(z)). \tag{1.5}$$

Note that the order of the operators $z^{-\alpha_j}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_j}z^{\alpha_j+n_j}$, $j = 1, \ldots, r$, in the product is not important since they are commuting [proposition 3, 4]. The main result of this paper is the existence of a *lowering operator*.

Theorem 1.1. *The multiple orthogonal polynomials of table 1 satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}z}P_{\vec{n}}^{\vec{\alpha};\beta}(z) = \sum_{j=1}^{r} c_{\vec{n};j}^{\vec{\alpha};\beta}P_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j;\beta+\nu}(z) \tag{1.6}$$

with

$$c_{\vec{n};j}^{\vec{\alpha};\beta} = A\left(\alpha_j + \sum_{i=1}^{r}n_i - 1, \beta\right)\frac{\prod_{i=1}^{r}(\alpha_i - \alpha_j + n_i)}{\prod_{i=1,i\neq j}^{r}(\alpha_i - \alpha_j)}, \qquad j = 1, \ldots, r. \tag{1.7}$$

As a corollary of the raising and lowering operators we then easily recover a linear differential equation of order $r + 1$ for the multiple orthogonal polynomials $P_{\vec{n}}^{\vec{\alpha};\beta}$ of table 1.

Theorem 1.2. *For the multiple orthogonal polynomials $P_{\vec{n}}^{\vec{\alpha};\beta}$ of table 1 we have*

$$\begin{aligned}
&\left[\prod_{i=1}^{r}\left(z^{-\alpha_i}\frac{\mathrm{d}}{\mathrm{d}z}z^{\alpha_i+1}\right)\right]\left(\omega^{\beta+\nu}(z)\frac{\mathrm{d}}{\mathrm{d}z}P_{\vec{n}}^{\vec{\alpha};\beta}(z)\right)\\
&\quad= \sum_{j=1}^{r}c_{\vec{n};j}^{\vec{\alpha};\beta}\left[\prod_{i=1,i\neq j}^{r}\left(z^{-\alpha_i}\frac{\mathrm{d}}{\mathrm{d}z}z^{\alpha_i+1}\right)\right]\left(\omega^{\beta}(z)P_{\vec{n}}^{\vec{\alpha};\beta}(z)\right),
\end{aligned} \tag{1.8}$$

where $c_{\vec{n};j}^{\vec{\alpha};\beta}$ are defined as in (1.7).

Proof. Apply the operators $\left[\prod_{i=1}^{r}\left(z^{-\alpha_i}\frac{\mathrm{d}}{\mathrm{d}z}z^{\alpha_i+1}\right)\right]\omega^{\beta+\nu}(z)$ to (1.6). Then note that by the raising operators (1.4)

$$\begin{aligned}
&\left[\prod_{i=1}^{r}\left(z^{-\alpha_i}\frac{\mathrm{d}}{\mathrm{d}z}z^{\alpha_i+1}\right)\right]\left(\omega^{\beta+\nu}(z)P_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j;\beta+\nu}(z)\right)\\
&\quad= \left[\prod_{i=1,i\neq j}^{r}\left(z^{-\alpha_i}\frac{\mathrm{d}}{\mathrm{d}z}z^{\alpha_i+1}\right)\right]\left(\omega^{\beta}(z)P_{\vec{n}}^{\vec{\alpha};\beta}(z)\right),
\end{aligned}$$

because of the commutativity of the operators $z^{-\alpha_i}\frac{\mathrm{d}}{\mathrm{d}z}z^{\alpha_i+1}$, $i = 1, \ldots, r$, [proposition 3, 4]. □

1.2. Table 2

In these cases we consider r weights of the same classical two-parameter family with different exponential rates at infinity. In particular they have the form $w^{\alpha,\beta_j}(z) = \upsilon^{\alpha}(z)\,\mathrm{e}^{\beta_j z}$, $j = 1,\dots,r$, with $\beta_j - \beta_k \notin \mathbb{Z}$, $j \neq k$. As mentioned in, e.g., [14], the corresponding multiple orthogonal polynomials $P_{\vec{n}}^{\alpha;\vec{\beta}}$ again have r *raising operators*. Note that $\nu = 0$ in these cases. Using integration by parts and the properties (P1)–(P5) one can then easily prove

$$\frac{1}{w^{\alpha,\beta_j}(z)}\frac{\mathrm{d}}{\mathrm{d}z}\left(w^{\alpha+\mu,\beta_j}(z)P_{\vec{n}-\vec{e}_j}^{\alpha+\mu;\vec{\beta}}(z)\right) = P_{\vec{n}}^{\alpha;\vec{\beta}}(z), \qquad n_j > 0, \tag{1.9}$$

$j = 1,\dots,r$, where the correct degree is provided by (P3). As before relations (1.9) fix the normalization of the polynomials $P_{\vec{n}}^{\alpha;\vec{\beta}}$. Repeatedly applying this raising operators gives the *Rodrigues formula*

$$P_{\vec{n}}^{\alpha;\vec{\beta}}(z) = \frac{1}{\upsilon^{\alpha}(z)}\left[\prod_{j=1}^{r}\left(\mathrm{e}^{-\beta_j z}\frac{\mathrm{d}^{n_j}}{\mathrm{d}z^{n_j}}\,\mathrm{e}^{\beta_j z}\right)\right](\upsilon^{\alpha+\mu|\vec{n}|}(z)). \tag{1.10}$$

Here the product of the r differential operators $\mathrm{e}^{-\beta_j z}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n_j}\mathrm{e}^{\beta_j z}$, $j = 1,\dots,r$, can be taken in any order since they are commuting [proposition 4, 4]. In this paper we prove the following *lowering operator* for these polynomials.

Theorem 1.3. *The multiple orthogonal polynomials of table 2 satisfy*

$$\frac{\mathrm{d}}{\mathrm{d}z}P_{\vec{n}}^{\alpha;\vec{\beta}}(z) = \sum_{j=1}^{r} c_{\vec{n};j}^{\alpha;\vec{\beta}}P_{\vec{n}-\vec{e}_j}^{\alpha+\mu;\vec{\beta}}(z) \tag{1.11}$$

with $c_{\vec{n};j}^{\alpha;\vec{\beta}} = A(\alpha,\beta_j)n_j$, $j = 1,\dots,r$.

Combining the raising operators (1.9) and the lowering operator (1.11) we then easily find a linear differential equation of order $r+1$ for the polynomials $P_{\vec{n}}^{\alpha;\vec{\beta}}$ of table 2.

Theorem 1.4. *For the multiple orthogonal polynomials* $P_{\vec{n}}^{\alpha;\vec{\beta}}$ *of table 2 we have*

$$\begin{aligned}&\left[\prod_{i=1}^{r}\left(\mathrm{e}^{-\beta_j z}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\beta_j z}\right)\right]\left(\upsilon^{\alpha+\mu}(z)\frac{\mathrm{d}}{\mathrm{d}z}P_{\vec{n}}^{\alpha;\vec{\beta}}(z)\right)\\ &\qquad = \sum_{j=1}^{r} c_{\vec{n};j}^{\alpha;\vec{\beta}}\left[\prod_{i=1,i\neq j}^{r}\left(\mathrm{e}^{-\beta_j z}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\beta_j z}\right)\right]\left(\upsilon^{\alpha}(z)P_{\vec{n}}^{\alpha;\vec{\beta}}(z)\right),\end{aligned} \tag{1.12}$$

where the $c_{\vec{n};j}^{\alpha;\vec{\beta}}$ *are defined as in theorem 1.3.*

Proof. First apply $\left[\prod_{i=1}^{r}\left(\mathrm{e}^{-\beta_j z}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\beta_j z}\right)\right]\upsilon^{\alpha+\mu}(z)$ to (1.11). Then note that the operators $\mathrm{e}^{-\beta_j z}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\beta_j z}$, $j = 1,\dots,r$, commute [proposition 4, 4]. So, by the raising operators (1.9),

$$\left[\prod_{i=1}^{r}\left(\mathrm{e}^{-\beta_j z}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\beta_j z}\right)\right]\left(\upsilon^{\alpha+\mu}(z)P_{\vec{n}-\vec{e}_j}^{\alpha+\mu;\vec{\beta}}(z)\right) = \left[\prod_{i=1,i\neq j}^{r}\left(\mathrm{e}^{-\beta_j z}\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{e}^{\beta_j z}\right)\right]\left(\upsilon^{\alpha}(z)P_{\vec{n}}^{\vec{\alpha};\beta}(z)\right).$$

$\square$

2. Proofs of theorem 1.1 and theorem 1.3

In order to prove theorem 1.1 we need the following technical lemma.

Lemma 2.1. *For $r \geqslant 2$ different values $\alpha_1, \ldots, \alpha_r$, and every multi-index $\vec{n} \in \mathbb{N}_0^r$ we have*

$$\sum_{\ell=1,\ell\neq j}^{r} \frac{\prod_{i=1,i\neq k}^{r}(\alpha_i - \alpha_\ell + n_i)}{\prod_{i=1,i\neq j,\ell}^{r}(\alpha_i - \alpha_\ell)} = \alpha_j - \alpha_k + \sum_{i=1,i\neq k}^{r} n_i, \qquad 1 \leqslant j,k \leqslant r. \tag{2.1}$$

Proof. Define the rational function

$$R(z) := -\frac{\prod_{i=1,i\neq k}^{r}(\alpha_i - z + n_i)}{\prod_{i=1,i\neq j}^{r}(\alpha_i - z)}.$$

Since all the α_ℓ are different and $\lim_{z\to\infty} R(z) = -1$ this function R can be written as

$$R(z) = -1 + \sum_{\ell=1,\ell\neq j}^{r} \frac{\operatorname{Res}(R,\alpha_\ell)}{z-\alpha_\ell}.$$

A simple calculation then gives

$$\begin{aligned}\sum_{\ell=1,\ell\neq j}^{r} \operatorname{Res}(R,\alpha_\ell) &= \lim_{z\to\infty} z(1+R(z)) \\ &= \lim_{z\to\infty} z\frac{\left(\sum_{i=1,i\neq k}^{r}(\alpha_i+n_i) - \sum_{i=1,i\neq j}^{r}\alpha_i\right)z^{r-2} + o(z^{r-2})}{\prod_{i=1,i\neq j}^{r}(z-\alpha_i)} \\ &= \alpha_j - \alpha_k + \sum_{i=1,i\neq k}^{r} n_i.\end{aligned}$$

Finally observe that the left-hand side is equal to the left-hand side of (2.1) by the definition of residuals. □

Proof of theorem 1.1. Using integration by parts and the properties (P1), (P4) and (P5) it is easily seen that $\frac{\mathrm{d}}{\mathrm{d}z}P_{\vec{n}}^{\vec{\alpha};\beta}(z)$ is a polynomial of degree $|\vec{n}|-1$ which is orthogonal to $1, z, \ldots, z^{n_j-2}$ with respect to the weight $w^{\alpha_j+1,\beta+\nu}$ on Γ, $j = 1, \ldots, r$. Since the weights have the form (1.3), the polynomials

$$P_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j;\beta+\nu}, \qquad j = 1, \ldots, r,$$

also satisfy these properties. Moreover, they form a basis for this subspace of polynomials. Indeed, these polynomials are linearly independent since if we have

$$\sum_{j=1}^{r} d_j P_{\vec{n}-\vec{e}_j}^{\vec{\alpha}+\vec{e}_j;\beta+\nu}(z) = 0,$$

then integrating with respect to $w^{\alpha_k,\beta+\nu}$ on Γ (and recalling that every multi-index is normal) gives $d_k = 0$, and this holds for $k = 1, \ldots, r$. We may therefore conclude that there exist constants $c_{\vec{n};j}^{\vec{\alpha};\beta}$ such that (1.6) holds.

We now assume that $n_j \geqslant 1$, $j = 1, \ldots, r$. Otherwise the value of r can be decreased in (1.6). If we integrate (1.6) with respect to $z^{n_k}w^{\alpha_k,\beta+\nu}$ on Γ, $k = 1, \ldots, r$, then by Cramer's rule we get

$$c_{\vec{n};j}^{\vec{\alpha};\beta} = \frac{\det M_j(\vec{\alpha};\beta;\vec{n})}{\det M(\vec{\alpha};\beta;\vec{n})}, \qquad j = 1, \ldots, r, \tag{2.2}$$

where for $1 \leqslant k, \ell, \leqslant r$,

$$[M(\vec{\alpha};\beta;\vec{n})]_{k,\ell} = \int_\Gamma P_{\vec{n}-\vec{e}_\ell}^{\vec{\alpha}+\vec{e}_\ell;\beta+\nu}(z) z^{n_k} w^{\alpha_k,\beta+\nu}(z)\,\mathrm{d}z,$$

$$[M_j(\vec{\alpha};\beta;\vec{n})]_{k,\ell} = \begin{cases} \displaystyle\int_\Gamma P_{\vec{n}-\vec{e}_\ell}^{\vec{\alpha}+\vec{e}_\ell;\beta+\nu}(z) z^{n_k} w^{\alpha_k,\beta+\nu}(z)\,\mathrm{d}z, & \ell \neq j, \\ \displaystyle\int_\Gamma \frac{\mathrm{d}}{\mathrm{d}z}\big(P_{\vec{n}}^{\vec{\alpha};\beta}(z)\big) z^{n_k} w^{\alpha_k,\beta+\nu}(z)\,\mathrm{d}z, & \ell = j. \end{cases}$$

Using integration by parts together with properties (P1)–(P5), the orthogonality conditions of $P_{\vec{n}}^{\vec{\alpha};\beta}$ and finally the raising operators (1.4) we obtain

$$\int_\Gamma \frac{\mathrm{d}}{\mathrm{d}z}\big(P_{\vec{n}}^{\vec{\alpha};\beta}(z)\big) z^{n_k} w^{\alpha_k,\beta+\nu}(z)\,\mathrm{d}z = -A(\alpha_k+n_k-1,\beta)\int_\Gamma P_{\vec{n}}^{\vec{\alpha};\beta}(z) z^{n_k} w^{\alpha_k,\beta}(z)\,\mathrm{d}z$$
$$= (-1)^{r+1} A(\alpha_k+n_k-1,\beta)\prod_{i=1}^r(\alpha_k-\alpha_i+n_k) I_k$$

with

$$I_k = \int_\Gamma P_{\vec{n}-\vec{e}}^{\vec{\alpha}+\vec{e};\beta+\nu r}(z) z^{n_k} w^{\alpha_k,\beta+\nu r}(z)\,\mathrm{d}z.$$

In a same manner we get

$$\int_\Gamma P_{\vec{n}-\vec{e}_\ell}^{\vec{\alpha}+\vec{e}_\ell;\beta+\nu}(z) z^{n_k} w^{\alpha_k,\beta+\nu}(z)\,\mathrm{d}z = (-1)^{r-1}\prod_{i=1,i\neq\ell}^r(\alpha_k-\alpha_i+n_k) I_k.$$

If we plug these two results in (2.2), then we see that $(-1)^{r+1}(\alpha_k-\alpha_j+n_k)I_k$ and $(-1)^{r+1}I_k$ are common factors in the kth row of $M_j(\vec{\alpha};\beta;\vec{n})$ and $M(\vec{\alpha};\beta;\vec{n})$, respectively. This leads to

$$c_{\vec{n};j}^{\vec{\alpha};\beta} = \frac{\prod_{k=1}^r(\alpha_k-\alpha_j+n_k)\det C_j(\vec{\alpha};\beta;\vec{n})}{\det C(\vec{\alpha};\vec{n})}, \qquad j=1,\ldots,r, \tag{2.3}$$

where for $1 \leqslant k, \ell \leqslant r$,

$$[C(\vec{\alpha};\vec{n})]_{k,\ell} = \prod_{i=1,i\neq\ell}^r (\alpha_k-\alpha_i+n_k),$$

$$[C_j(\vec{\alpha};\beta;\vec{n})]_{k,\ell} = \begin{cases} \displaystyle\prod_{i=1,i\neq j,\ell}^r (\alpha_k-\alpha_i+n_k), & \ell \neq j, \\ \displaystyle A(\alpha_k+n_k-1,\beta)\prod_{i=1,i\neq j}^r (\alpha_k-\alpha_i+n_k), & \ell = j. \end{cases}$$

We first study the denominator on the right-hand side of (2.3). Note that by subtracting the jth column from all the other columns in $C(\vec{\alpha};\vec{n})$,

$$\det C(\vec{\alpha};\vec{n}) = \prod_{\ell=1,\ell\neq j}^r (\alpha_\ell-\alpha_j)\det D_j(\vec{\alpha};\vec{n}) \tag{2.4}$$

with, for $1 \leqslant k, \ell \leqslant r$,

$$[D_j(\vec{\alpha};\beta;\vec{n})]_{k,\ell} = \begin{cases} \displaystyle\prod_{i=1,i\neq j,\ell}^r (\alpha_k-\alpha_i+n_k), & \ell \neq j, \\ \displaystyle\prod_{i=1,i\neq j}^r (\alpha_k-\alpha_i+n_k), & \ell = j. \end{cases}$$

Next, we have a closer look at $\det C_j(\vec{\alpha}; \beta; \vec{n})$. Add the ℓth column, $\ell = 1, \ldots, r$, $\ell \neq j$, multiplied by

$$\frac{\prod_{i=1}^{r}(\alpha_i - \alpha_\ell + n_i)}{\prod_{i=1, i\neq j,\ell}^{r}(\alpha_i - \alpha_\ell)} \delta_{\deg(\phi),2}$$

to the jth column. Then we apply lemma 2.1 and observe that for table 1 the property $A(\alpha, \beta) + x\delta_{\deg(\phi),2} = A(\alpha + x, \beta)$, $x \in \mathbb{R}$, holds. We finally get

$$\det C_j(\vec{\alpha}; \beta; \vec{n}) = A\left(\alpha_j + \sum_{i=1}^{r} n_i - 1, \beta\right) \det D_j(\vec{\alpha}; \vec{n}). \tag{2.5}$$

Combining (2.3), (2.4) and (2.5) then proves (1.7). □

Proof of theorem 1.3. Note that $\nu = 0$ for table 2. So, using integration by parts and the properties (P1), (P4) and (P5) one easily sees that $\frac{\mathrm{d}}{\mathrm{d}z} P_{\vec{n}}^{\alpha;\vec{\beta}}(z)$ is a polynomial of degree $|\vec{n}| - 1$ which is orthogonal to $1, z, \ldots, z^{n_j-2}$ with respect to the weight $w^{\alpha+\mu,\beta_j}$ on Γ, $j = 1, \ldots, r$. Note that the polynomials $P_{\vec{n}-\vec{e}_j}^{\alpha+\mu;\vec{\beta}}$, $j = 1, \ldots, r$, form a basis for the subspace of polynomials satisfying these conditions. Indeed, if

$$\sum_{j=1}^{r} d_j P_{\vec{n}-\vec{e}_j}^{\alpha+\mu;\beta}(z) = 0,$$

then integrating with respect to $z^{n_k-1} w^{\alpha+\mu,\beta_k}(z)$ on Γ gives $d_k = 0$, $k = 1, \ldots, r$, which means that they are linearly independent. (Here we use that every multi-index is normal.) So, there exist constants $c_{\vec{n};j}^{\alpha;\vec{\beta}}$ such that (1.11) holds.

Fix $j \in \{1, \ldots, r\}$. Integrating (1.11) with respect to $z^{n_j-1} w^{\alpha+\mu,\beta_j}(z)$ on Γ gives

$$c_{\vec{n};j}^{\alpha;\vec{\beta}} = \frac{\int_\Gamma \frac{\mathrm{d}}{\mathrm{d}z}\big(P_{\vec{n}}^{\alpha;\vec{\beta}}(z)\big) z^{n_j-1} w^{\alpha+\mu,\beta_j}(z)\,\mathrm{d}z}{\int_\Gamma P_{\vec{n}-\vec{e}_j}^{\alpha+\mu;\vec{\beta}}(z) z^{n_j-1} w^{\alpha+\mu,\beta_j}(z)\,\mathrm{d}z}, \qquad n_j \geqslant 1. \tag{2.6}$$

Using integration by parts together with properties (P1)–(P5) and the orthogonality conditions of $P_{\vec{n}}^{\alpha;\vec{\beta}}$, for the numerator we obtain

$$\int_\Gamma \frac{\mathrm{d}}{\mathrm{d}z}\big(P_{\vec{n}}^{\alpha;\vec{\beta}}(z)\big) z^{n_j-1} w^{\alpha+\mu,\beta_j}(z)\,\mathrm{d}z = -A(\alpha, \beta_j) \int_\Gamma P_{\vec{n}}^{\alpha;\vec{\beta}}(z) z^{n_j} w^{\alpha,\beta_j}(z)\,\mathrm{d}z. \tag{2.7}$$

Next, we can apply the raising operator (1.9) and again integration by parts in order to get

$$\int_\Gamma P_{\vec{n}}^{\alpha;\vec{\beta}}(z) z^{n_j} w^{\alpha,\beta_j}(z)\,\mathrm{d}z = -n_j \int_\Gamma P_{\vec{n}-\vec{e}_j}^{\alpha+\mu;\vec{\beta}}(z) z^{n_j-1} w^{\alpha+\mu,\beta_j}(z)\,\mathrm{d}z. \tag{2.8}$$

Combining (2.6), (2.7) and (2.8) then completes the proof. □

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